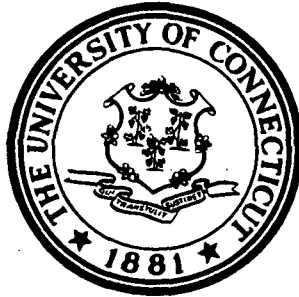


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**SURVEY OF ADAPTIVE CONTROL USING LIAPUNOV DESIGN**

**D. P. Lindorff  
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**Technical Report 72-7**

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# SURVEY OF ADAPTIVE CONTROL USING LIAPUNOV DESIGN

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## I. Introduction

This paper surveys the literature devoted to the synthesis of model-tracking adaptive systems based on application of Liapunov's second method. In the early work [B1, D1,2,3, H1, O1, P4, W1,2], the model tracking problem was approached by using the sensitivity or gradient approach, without assurance of global asymptotic stability. RANG [R1], SHACKCLOTH and BUTCHART [S1,2] and PARKS [P1], were first to employ the Liapunov design in finding an adaptive control law which guaranteed global stability. It is the purpose here to introduce the basic synthesis procedure, and to critically review extensions to the theory which have appeared since 1966, relating to: design for relative stability, reduction of order techniques, design with disturbance, design with time variable parameters, multivariable systems, identification, and an adaptive observer.

## II. Synthesis Using Liapunov's Second Method

The basic problem to be considered in this survey is that of designing a model tracking system for stability without specifying exact values of the plant parameters. Liapunov's stability theorems offer a means of synthesizing various control laws which offer possible solutions to this design problem, the particular solution depending in part upon the form of the Liapunov function selected. In this section the rudimentary ideas involved will be introduced in a somewhat limited context.

The concept which is central to adaptive schemes to be discussed in this survey can be explained with reference to the model tracking system in Fig. 2.1, for which the state equation of the stable model is given by

$$\dot{\underline{y}} = \underline{A}\underline{y} + \underline{B}\underline{r} \quad (\text{model}) \quad \underline{y}(0) = \underline{y}^0 \quad (2.1)$$

and that of the time-invariant plant by

$$\dot{\underline{x}} = \underline{A}^* \underline{x} + \underline{B}^* \underline{u}. \quad (\text{plant}) \quad \underline{x}(0) = \underline{x}^0. \quad (2.2)$$

Here  $\underline{y} = \{y_i\}$ ,  $\underline{x} = \{x_i\}$  are  $n$  dimensional state vectors, and  $\underline{r}, \underline{u}$  are  $m$  dimensional control inputs.  $\underline{A}^*, \underline{B}^*$  contain unknown coefficients. If the differential equation of the tracking error ( $\underline{e} = \underline{y} - \underline{x}$ ) is now written in the form

$$\dot{\underline{e}} = \underline{A}\underline{e} + \underline{f} \quad (2.3)$$

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where  $\underline{f} = (\underline{A} - \underline{A}^*)\underline{x} + \underline{B}\underline{r} - \underline{B}^*\underline{u}$ , then the control objective is to manipulate  $\underline{f}$  in some way so that  $\lim_{t \rightarrow \infty} \underline{e}(t) = \underline{0}$ . To this end we introduce the positive definite function

$$V = \frac{1}{2}(\underline{e}^T \underline{P} \underline{e} + \underline{h}(\underline{\phi}, \underline{\psi})) \quad (2.4)$$

where  $\underline{\phi}, \underline{\psi}$  are matrices of parameter vectors  $\underline{\phi}_i, \underline{\psi}_i$  to be defined. Then along the trajectory of (2.3) we obtain for the time derivative of  $V$

$$\dot{V} = -\underline{e}^T \underline{Q} \underline{e} + \underline{e}^T \underline{P} \underline{f} + \dot{\underline{h}} \quad (2.5)$$

where

$$-\underline{Q} = \underline{A}^T \underline{P} + \underline{P} \underline{A}. \quad (2.6)$$

By a theorem of Liapunov [L1], with any  $\underline{Q}^T = \underline{Q} > 0$ , it follows that  $\underline{P} = \underline{P}^T > 0$  is a unique solution to (2.6) iff  $\underline{A}$  is a stability matrix, as assumed.

Using the classification suggested by PHILLIPSON [P2], we will introduce two methods which have been reported for causing  $\underline{e} \rightarrow 0$ .

### Input Modification

This scheme uses the solution obtained with  $\underline{h} \equiv 0$ . Although it is not in itself useful in synthesizing an adaptive control law, i. e. one in which a set of parameters are automatically adjusted so as to reduce the tracking error, it has been used to advantage in solving certain of the design problems to be discussed in the survey.

With  $\underline{h} \equiv 0$ , asymptotic stability in  $\underline{e}$  will be satisfied if [G3]:

$$\underline{e}^T \underline{P} \underline{f} = 0 \text{ for } \underline{A} - \underline{A}^* = 0 \\ \leq 0 \text{ otherwise.} \quad (2.7)$$

This inequality cannot in general be satisfied. However in some restricted cases, such as if (2.1), (2.2) are in phase variable form, a solution exists.

### Feedback Synthesis

This scheme differs from input modification in that parameters in the system are adjusted continuously so that in the simplified case treated here  $\underline{e} \rightarrow 0$ .

In feedback synthesis, we write for (2.4)

$$V = \frac{1}{2}(\underline{e}^T \underline{P} \underline{e} + \sum_{i=1}^n \underline{\phi}_i^T \underline{\phi}_i + \sum_{i=1}^m \underline{\psi}_i^T \underline{\psi}_i) \quad (2.8)$$

wherein  $\underline{\phi}_i, \underline{\psi}_i$  are misalignment parameter vectors

to be defined. In this case (2.5) becomes

$$\dot{V} = -\underline{e}^T Q \underline{e} + \underline{e}^T P \underline{f} + \sum_{i=1}^n \dot{\phi}_i^T \phi_i + \sum_{i=1}^m \dot{\psi}_i^T \psi_i. \quad (2.9)$$

Let  $E$  define the whole state space with  $\xi \in E$ , where  $\xi$  is defined by  $\xi^T = [\underline{e}_1^T, \phi_1^T, \dots, \phi_n^T, \psi_1^T, \dots, \psi_m^T]$ . Let  $E_1 \subset E$ , where  $E_1$  is the  $n$  dimensional space with  $\underline{e} \in E_1$ . Clearly  $V$  in (2.8) is positive definite in  $E$ .

The basic idea in feedback synthesis is to specify  $(\phi_1, \psi_1)$  in (2.9) so that

$$\underline{e}^T P \underline{f} + \sum_{i=1}^n \dot{\phi}_i^T \phi_i + \sum_{i=1}^m \dot{\psi}_i^T \psi_i \equiv 0, \quad (2.10)$$

and consequently

$$\dot{V} = -\underline{e}^T Q \underline{e}. \quad (2.11)$$

Since  $\dot{V}$  is only negative semidefinite in  $E$ , and negative definite in  $E_1$ , we may conclude from the theorems of Liapunov [L1] that the equilibrium at  $\underline{e}=0$  is asymptotically stable, and the equilibrium at  $\xi=0$  is stable. It follows in the present formulation that  $\underline{e} \rightarrow 0$ , and that the missalignment parameter vectors are bounded. Using a theorem of Lasalle's it can be shown that  $\xi \rightarrow 0$  if the input  $\underline{r}$  is periodic and of rich enough frequency content [L2].

In implementing the controls to satisfy (2.10), there are two schemes which will be described in this survey as direct and indirect adaptation.

#### Direct Adaptation

In this scheme  $\underline{u} = \underline{r}$  in (2.2).  $\phi$  and  $\psi$  are in turn defined by  $\phi = A-A^*$ ,  $\psi = B-B^*$ , so that (2.3) becomes

$$\begin{aligned} \dot{\underline{e}} &= A \underline{e} + \underline{f} \\ &= A \underline{e} + \phi \underline{x} + \psi \underline{r}. \end{aligned} \quad (2.12)$$

(2.10) can be satisfied if

$$\begin{aligned} \dot{\phi}_i^T &= -\underline{e}^T P x_i, \quad (i=1, \dots, n) \\ \dot{\psi}_i^T &= -\underline{e}^T P r_i, \quad (i=1, \dots, m). \end{aligned} \quad (2.13)$$

If adaptive control (Fig. 2.1) is the objective, the elements of  $A^*$ ,  $B^*$  should be adjusted, in which case (2.13) becomes

$$\begin{aligned} \dot{a}_i^* &= \underline{e}^T P x_i, \quad i=1, \dots, n \\ \dot{b}_i^* &= \underline{e}^T P r_i, \quad i=1, \dots, m. \end{aligned} \quad (2.14)$$

#### Indirect Adaptation

Assuming that the elements of  $A^*$ ,  $B^*$  are not directly adjustable it becomes necessary to modify the system. Consider the single-input single-output plant defined by

$$\frac{x_1}{u} = \frac{k^*}{s^n + a_n^* s^{n-1} + \dots + a_1^*}$$

By introducing  $k_r, k_1, k_2, \dots, k_n$  as adjustable parameters as in Fig. 2.2, it is seen that the compensated system has been cast into the form

$$\dot{\underline{x}} = A^* \underline{x} + \underline{b}^* \underline{r}$$

where

$$A^* = \begin{bmatrix} 0 & \vdots & I \\ \vdots & \ddots & \vdots \\ a_{n1}^* & \dots & a_{nn}^* \end{bmatrix}, \quad \underline{b}^{*T} = [0, \dots, 0, b_n^*],$$

with  $b_n^* = k_r k^*$ , and  $a_{ni}^* = -a_i^* + k_i, i=1, \dots, n$ . Thereby the problem is amenable to direct adaptation by application of (2.14). For the single-input case it is noted that only  $n$  coefficients in  $A^*$  can be adjusted. The multivariable problem suffers restrictions also [L3].

### III. Degree of Stability - Improved Speed of Response

The adaptive step response [S1] has been helpful in analyzing the relative stability of adaptive systems. With the aid of this concept PHILLIPSON [P2] showed that it is possible in the case of an adaptive gain to improve the relative stability by appropriately modifying the adaptive control law. GILBERT and MONOPOLI [G1] formalized the synthesis procedure by redefining the  $V$  function in (2.8).

In the ensuing discussion let the error equation be defined by

$$\dot{\underline{e}} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \underline{e} + \begin{bmatrix} 0 \\ k - k^* \end{bmatrix} \underline{r} \quad (3.1)$$

where  $k^*$  is the only adjustable parameter. If (2.8) is written as

$$V = \frac{1}{2} (\underline{e}^T P \underline{e} + \frac{\psi^2}{h_1}), \quad h_1 = \text{const}. \quad (3.2)$$

then (2.5) becomes

$$\dot{V} = -\underline{e}^T Q \underline{e} + \underline{e}^T P_2 (k - k^*) \underline{r} + \frac{\dot{\psi} \psi}{h_1} \quad (3.3)$$

wherein  $P_2^T = [p_{12} \ p_{22}]$ , and  $P$  is found according to (2.6). In this case, with  $\psi = k - k^*$ , it is seen that (2.10) is satisfied if

$$\dot{k}^* = (p_{12} e_1 + p_{22} e_2) h_1 \underline{r}. \quad (3.4)$$

If  $\underline{r}$  is assumed to be a step of magnitude  $R$ , then differentiation of (3.1) leads to the characteristic equation

$$s^3 + a_1 s^2 + (K_1 s + K_2) R^2 = 0. \quad (3.5)$$

The roots of this equation characterize the adaptive step response. A typical root locus, as shown in Fig. 3.1, demonstrates that the relative stability is degraded with increasing  $R^2$ , even though the system by design cannot be unstable.

As a means of improving the relative stability with increasing  $R^2$  Phillipson modified the

adaptive rule in (3.4) by setting  $u = k^*r + u_1$ , and using  $u_1$  to insert input modification. Thus, with  $u_1 = \gamma k^*r$ , (2.5) becomes

$$\dot{V} = -\underline{e}^T Q \underline{e} - (\underline{e}^T P_2)^2 h_1 r^2. \quad (3.6)$$

The effect has been to make  $\dot{V}$  more negative with  $r^2$ . It is also instructive to examine the adaptive step response. The equation corresponding to (3.5) is

$$s^3 + a_1 s^2 + a_0 s + (p_{12} + p_{22} s)(1 + \gamma s) h_1 r^2 = 0. \quad (3.7)$$

The root locus indicates an improved relative stability for large  $R^2$  (see Fig. 3.2).

More in the spirit of involving the Liapunov function in the synthesis procedure, GILBART and MONOPOLI [G1] have proposed modifying the  $V$  function in (2.8) so that the desired result is obtained more routinely. The concept will be illustrated using the system equation (3.1). According to [G1] we introduce the modified  $V$  function

$$V = \frac{1}{2} \underline{e}^T P \underline{e} + \frac{1}{2} \frac{(\psi + h_1 z)^2}{h_2}. \quad (3.8)$$

The form for (2.5) now becomes

$$\dot{V} = -\underline{e}^T Q \underline{e} + \underline{e}^T P_2 (k - k^*)r + \frac{1}{h_2} (\dot{\psi} + h_1 \dot{z} + h_1 z \dot{z} + h_1^2 z \dot{z}). \quad (3.9)$$

If we equate for this example

$$z = \underline{e}^T P_2 r \quad (3.10)$$

$$\psi = k - k^*$$

$$\dot{\psi} = -h_2 z - h_1 \dot{z}$$

then (3.9) becomes

$$\dot{V} = -\underline{e}^T Q \underline{e} - h_1 (\underline{e}^T P_2 r)^2. \quad (3.11)$$

This is seen to be similar to (3.6). For the adaptive control, we have from (3.10)

$$k^* = h_1 \underline{e}^T P_2 r + h_2 \int_{t_0}^t \underline{e}^T P_2 r dt + k(t_0). \quad (3.12)$$

The general matrix formulation of this scheme for the entire parameter set  $\phi, \psi$  is given in [G2], together with simulation results showing that improvement in convergence time can be obtained.

#### IV. Reduction of Order

The basic model-reference Liapunov adaptive law requires measurement of the entire error vector for its implementation. If all the state variables of the plant under control are not available, then the basic adaptive law is inadequate since ignorance of parameters and inherent system noise may

prohibit their generation by an observer or by differentiators.

To overcome this practical problem investigators have attempted to find adaptive laws that require a minimum of state variable measurements. The first such attempt by PARKS [P1] was for the restrictive system shown in Figure 4.1 in which only the constant plant input gain  $K_V$  is mismatched to model gain  $K$ . The adaptive law adjusts  $K_d$  so that the error vanishes. Parks use of KALMAN lemma [K1] subsequently extended by MONOPOLI [M3] using the KALMAN-MEYER lemma [M1], has shown that the adaptive law  $\dot{K}_c = h_1 e_1 r$ , in which only the output error  $e_1$  appears, is sufficient for asymptotic stability of  $\underline{e}$  if

- 1)  $\frac{N(s)}{D(s)}$  is a positive real function (4.1)
- 2)  $[1 \ 0 \ 0 \ \dots \ 0] A b \neq 0$

where  $\frac{N(s)}{D(s)}$  is the transfer function representation of the plant equation  $\dot{x} = Ax + br$ . If (2) does not hold but (1) does, the set  $\dot{V} = 0$  must be examined to insure asymptotic stability of  $\underline{e}$ .

MONOPOLI [M3] extended the criteria to non-positive real transfer function that can be made positive real by multiplying by a polynomial in  $s$  with root of negative real parts. In doing so, the resulting adaptive law requires  $n-m-2$  derivatives of the output error  $e_1$  where  $n$  is the number of plant poles and  $m$  is the number of plant zeroes.

In the more general case where adaptation of both plant poles and zeroes are desired, the basic indirect adaptive scheme fails (but not direct adaptation) even with full state measurement available. This is seen since, referring to Figure 4.2 with  $D(s) = 1$ , the error equation

$$N(s)e_1 = [Q(s) - N(s) + P(s) H(s)]x_1 + [M(s) - P(s) G(s)]r \quad (4.2)$$

reveals  $H(s)$ , a polynomial, may have at most only  $n-m$  free parameters so as not to increase the order of the  $n$ th order polynomial  $Q - N + PH$ ; consequently there exists no steady-state solution to the problem of adjusting  $n$  parameters in  $Q - N + PH$  using state feedback.

MONOPOLI and GILBART [M4], however, have employed dynamic feedback on a reduced state of order  $n-m-1$  to accomplish adaptation with plant zeroes. Moreover, they have shown that the plant matrix  $A^*$  need not be of any particular form; nor must  $A^*$  be stable nor  $C^*(sI - A^*)^{-1}B^*$  be positive real; and the output matrix  $C^*$  need not be known.

Their basic idea is for  $D(s)$ , after adaptation, to cancel  $P(s)$  with the model zeroes placed in cascade with the plant by  $G(s)$ . Due to the cancellation, the zeroes of the plant necessarily must be in the LHP.

The vector error equation (2.12) is

"collapsed" to yield a scalar error equation in the output error  $e_1$  from which the synthesis proceeds:

$$e_1^{(n)} + \sum_{i=1}^{n-1} a_i e_1^{(i)} = \sum_{i=0}^{n-1} \Delta a_i x_1^{(i)} + \sum_{i=0}^m \Delta b_i r^{(i)} + \sum_{i=0}^m \hat{b}_i u^{(i)} \quad (4.3)$$

For clarity in illustrating the synthesis procedure, a 2<sup>nd</sup> order system with one zero will be treated here. For this system, (4.3) becomes

$$(s^2 + a_1 s + a_2) e_1 = (\Delta a_1 s + \Delta a_0) x_1 + \Delta b_0 r + (s + b_0) u \quad (4.4)$$

in which initial conditions have been ignored and  $\Delta b_1 = 0$  and  $b_1 = 1$  for simplicity in this treatment. Dividing (4.4) by  $s+c$ ,  $0 < c < a_1$ , and taking the Laplace inverse transformation yields

$$\begin{aligned} \dot{e}_1 + (a_1 - c) e_1 &= [c(a_1 - c) - a_2] \mathcal{L}^{-1} \left( \frac{e_1}{s+c} \right) + \Delta a_1 x_1 \\ &+ [\Delta a_0 - c \Delta a_1] \mathcal{L}^{-1} \left( \frac{x_1}{s+c} \right) + \Delta b_0 \mathcal{L}^{-1} \left( \frac{r}{s+c} \right) \\ &+ u + [b_0 - c] \mathcal{L}^{-1} \left( \frac{u}{s+c} \right). \end{aligned} \quad (4.5)$$

Now if  $u$  is chosen as  $u = \sum k_i v_i$ , where the  $v_i$  are each of the terms on the right side of (4.5) (i.e.  $v_1 = \mathcal{L}^{-1} \left( \frac{e_1}{s+c} \right)$ ,  $v_2 = x_1$ , etc.) then (4.5) has the form for which the indirect adaptation scheme can be applied with the important difference that the resulting adaptive law contains only  $e_1$  and not its derivatives.

In general, a division polynomial, such as that which produced (4.5), should be of order  $m$  with zeroes chosen so that the left side of (4.5) is stable. Then the left side contains  $n-m$  terms, so the resulting adaptive law contains  $n-m-1$  derivatives of  $e_1$ .

It should be pointed out that since vector state information is destroyed in collapsing the error equation to (4.3), convergence of  $e_1$  (output error) does not in general imply that the state variables converge. If the output matrix is known and  $A$  is in phase variable form, then convergence of  $e_1$  implies convergence of the first  $n-m-1$  state variables.

At present, indirect adaptation cannot take place when there is a right half-plane zero in the system.

#### V. Effects of Disturbance in Adaptive Control

As noted in Section II, if disturbance is neglected then, according to (2.11),  $\dot{V}$  becomes negative definite in  $E_1$  and negative semidefinite in  $E$ . However the input to the system can be chosen so that the solution  $\dot{V} = 0$  can be satisfied only at the origin of  $E$  space.

Recently LINDORFF [L4] and NARENDRA et al [N1] have considered the effect of disturbance upon stability. In [L4] it is shown that, even though

$\underline{e}$  remains bounded (theoretically), disturbance can cause the adaptive gains to be unstable (unbounded). In [N1] a modified scheme is derived such that  $\dot{V}$  is strictly negative in  $E$  outside of some bounded region about the origin, thereby guaranteeing boundedness in  $E$ .

In [L4] the effect of disturbance  $\underline{d}$  and incomplete adaptation has been examined with reference to the single-input plant described by the equation

$$\dot{x} = A^* x + b^*(r + u_1) + \underline{d}. \quad (5.1)$$

The tracking error in this case defined by

$$\underline{\dot{e}} = A \underline{e} + \underline{f}_1 + \underline{f}_2, \quad \underline{f}_1 + \underline{f}_2 = \underline{\dot{f}}$$

where  $\underline{f}_1$  includes all adjustable parameters

$\Phi, \Psi$  (although these need not be the entire set of unknown parameters), and  $\underline{f}_2$  contains the remaining terms in  $\underline{f}$ , including  $\underline{d}$ . The adaptive controls when applied to this problem cause  $\dot{V}$  to be reduced to the form

$$\dot{V} = -\underline{e}^T Q \underline{e} + \underline{e}^T P \underline{f}_2. \quad (5.2)$$

Since  $\dot{V}$  in (5.2) is strictly negative outside of some bounded region about  $\underline{e} = 0$  in  $E_1$ , and indefinite elsewhere,  $\underline{e}$  will ultimately lie inside a calculable region about the origin. However, due to the presence of  $\underline{f}_2$ ,  $\dot{V}$  is indefinite in  $E$ , and stability in  $E$  can no longer be guaranteed. This has been demonstrated for the case in which (5.1) is in phase variable form [L4].

NARENDRA et al [N1] have considered the problem of synthesizing an adaptive control law which guarantees boundedness of the errors when disturbance is present. Since their method also allows for time varying plant parameters, we shall include this degree of generality in the discussion, although the time variable case is treated in more detail in Section VI. For simplicity of exposition consider the first order differential equations

$$\begin{aligned} \text{(stable model)} \quad \dot{y} &= -ay + r \\ \text{(plant)} \quad \dot{x} &= -(a^* + k)x + r + d, \quad a^* = a^*(t) \end{aligned} \quad (5.6)$$

in which  $k$  is an adjustable parameter,  $d$  is a disturbance entering the plant, and  $r$  is the common input. The tracking error ( $e = y - x$ ) is governed by

$$\dot{e} = -ae + \phi x - d \quad (5.7)$$

with  $\phi = -a + a^* + k$ . If we now choose the positive definite form

$$V = \frac{1}{2} (\gamma e^2 + \phi^2), \quad \gamma > 0 \quad (5.8)$$

and introduce the modified adaptive control

$$\dot{k} = -\beta k - \gamma e x \quad (5.9)$$

the equation for  $\dot{V}$  becomes

$$\dot{V} = -\gamma a e^2 - \beta \phi^2 + (\beta(a^* - a) + \dot{a}^*) \phi - \gamma d e. \quad (5.10)$$

From this result it follows that boundedness in  $e$ ,  $\phi$  is guaranteed since  $\dot{V}$  contains a negative definite part in  $e$  and  $\phi$ . These quadratic terms control the sign of  $\dot{V}$  for large enough values of  $|e|$ ,  $|\phi|$ , if  $(a^* - a)$ ,  $\dot{a}^*$  are bounded. This result

in turn depends upon the introduction of  $\beta$  in the modified adaptive control law (5.9).

This design has been generalized [N1] for the single-input  $n^{\text{th}}$  order plant. Simulation results indicate that, in the absence of noise disturbance, best results are obtained with  $\beta = 0$ .

## VI Time-Varying Parameters

For the time-varying plant

$$\dot{x} = A^*(t)x + B^*(t)(r - u) \quad (6.1)$$

to behave as a time-invariant model requires an indirect adaptive law that depends upon the time derivative of an unavailable quantity if the synthesis of Section II is followed. PORTER and TATNALL [P3] have pointed out that this term may be ignored under some conditions on  $A^*(t)$  and  $B^*(t)$  for eventual asymptotic stability of  $e$ . These conditions are restrictive, however.

In the more general case, use of a new adaptive law due to NARENDRA, TRIPATHI, LUDERO and KUDVA [N2] leads to the determination of Lagrange stability bounds [N1]. To illustrate, consider a first-order plant (6.1) with  $B^*(t) = B = 1$ . The corresponding error equation with  $u = kx$  is

$$\dot{e} + ae = (a^* - a + k)x = \phi x. \quad (6.2)$$

Using the adaptive law [N2]

$$\dot{k} = \beta k - ex, \beta > 0 \quad (6.3)$$

the Liapunov function can be shown to be

$$V = \frac{1}{2} (e^2 + \phi^2) \quad (6.4)$$

$$\dot{V} = -ae^2 - \beta\phi^2 - \phi[a - a^*(t)]\beta + \dot{a}\phi$$

Bounds on the region of attraction can now be found by determining the smallest ellipse  $V=c$  that encloses the region of indefinite  $\dot{V}$ .

It is noted that (6.3) does not produce an asymptotically stable error even in time-invariant systems unless  $\beta=0$ .

A more extensive analysis along these lines is given by NARENDRA and TRIPATHI [N2].

MONOPOLI, GILBART, and THAYER [M2] produced a "practically asymptotically stable" system (see ref. [L1]) of Figure 3.1 when only the time-varying plant input gain  $K_v(t)$  is adapted. The plant must have the same general attributes as in Section III. They found, using a Liapunov function  $V = 1/2(\lambda \underline{w}^T \underline{P} \underline{w} + \lambda(t) \phi^2)$ , that practical asymptotic stability of  $e$  is attained with an adaptive law  $\dot{\underline{K}} = \lambda \underline{w}^T \underline{P} \underline{b} r + \delta K(K - K_v) |r|$  where  $\underline{w}$  is related to  $n-m-1$  derivatives of  $e$ ,  $n$  &  $m$  defined as in Section III. The region of attraction is inversely proportional to the magnitude of  $\delta$  and  $r$ .

The reduction of order with time-varying parameters was extended by GILBART and MONOPOLI [G3] to include adaptation of time-varying plant input coefficients and characteristic polynomial coefficients. A scalar error equation in  $e_1$  analogous to (4.3) but with time-varying coefficients on the right side is assumed. For clarity, a second order example is demonstrated here.

Suppose first that the second order plant is without input derivatives. Then the error equation is

$$\frac{1}{b_1(t)} (e_1 + a_2 \dot{e}_1 + a_1 e_1) = \frac{\Delta b_1(t)}{b_1(t)} r + \frac{\Delta a_2(t)}{b_1(t)} \dot{x} + \frac{\Delta a_1(t)}{b_1(t)} x - u, \hat{b}(t) > 0 \quad (6.5)$$

$$= [f_1 + g_1(t)]r + [f_2 + g_2(t)]\dot{x} + [f_3 + g_3(t)]x - u$$

in which the coefficients have been broken into time-invariant parts  $f$  and time-varying parts  $g$ . Choosing  $u = (k_1^f + k_1^g)r + k_2^f \dot{x} + k_3^f x$  and

$$\begin{aligned} \dot{k}_1^f &= \alpha_1 e^T P_2 r \\ \dot{k}_1^g &= -\alpha_1 e^T P_2 x^{(2-1)} \quad i=2,3 \end{aligned} \quad (6.6)$$

$$\dot{k}_1^g = M_1 \text{ sat} \left[ \frac{\beta_1}{M_1} e^T P_2 r \right]$$

$$\dot{k}_1^g = M_1 \text{ sat} \left[ \frac{\beta_1}{M_1} e^T P_2 x^{(2-1)} \right]_{i=2,3}$$

then the function  $V = e^T P e + \sum_{i=1}^3 (f_i + k_i^f)$

has a negative derivative outside a region whose boundary is directly proportional to the bound on  $|g_i(t)|$  and  $\hat{b}(t)$  and inversely proportional to  $\beta_i$ . Consequently (6.6) delivers strong practical stability of  $e$ .

It is shown also by GILBART and MONOPOLI [G3] that a reduction of order technique similar to that in Section IV, but modified to account for the time-varying nature of the scalar error equation (4.3), can be applied so that the resulting equation is in a form similar to (6.5) in order that adaptive laws like (6.6) may be used to produce a strong practically stable system. The reduction of order allows only  $n-m-1$  derivatives to be used in forming the adaptive laws.

## VII. Adaptive Control of Multivariable Systems

It has been noted by WINSOR and ROY [W2] that the adaptive control law in the form of (2.14) can be implemented for the general multivariable plant if the plant parameters are directly adjustable (direct adaptation). The practical case in which feedback control signals are used to implement compensation (indirect control) warrents attention, however, particularly for the multi-input systems.

Starting with (2.1), and (2.2), (2.3), LINDORFF [L3] has shown that certain conditions are imposed on the form of (2.1)(2.2) in order that the Liapunov design may yield a unique set of controls. This may be clarified by considering the term in (2.5)

$$\underline{e}^T P \underline{f} = \sum_{i=1}^n \underline{e}^T \underline{p}_i f_i \quad (7.1)$$

in which

$$\underline{f}_i = \sum_{j=1}^n \phi_{ij} x_j + \sum_{j=1}^m \psi_{ij} r + \sum_{j=1}^m b_{ij}^* u_j.$$

Since  $P$  is positive definite,  $\underline{p}_i$  and  $\underline{p}_j$  are linearly independent for all  $i \neq j$ . Therefore a different  $u_j$  must be identified with each  $\underline{f}_i$  in generating a particular component of the adaptive control law. It follows that not more than  $m$  components of  $\underline{f}$  can be nonzero, and that a stable adaptive control law can be realized if (1) there are no more outputs than inputs, (2) the state equation is written in partitioned phase variable form, (3) the matrices  $B^*$ ,  $B$  are in triangular form. Apparently no efforts to remove these restrictions have been reported. Extension of the reduction of order technique [M4] to the multivariable problem is one possibility which suggests itself (see Section IV).

### VIII. Identification

The identification problem can also be approached so that the process is inherently stable. LION [L5] has derived a very practical solution to the problem of identification for single-input, single-output plants, with guaranteed asymptotic stability. KUDVA and NARENDRA [K2] have applied the Liapunov synthesis method to the identification of time variable multivariable systems, illustrating an application of direct adaptation (Section II).

Lion uses the so called Generalized Equation Error System of Figure 8.1. In this scheme the parameters of  $\tilde{N}$ ,  $\tilde{D}$  are adjusted so as to minimize  $e^2$ . A significant feature of the method is that no derivatives of  $u, y$  are required ( $G$  is a low pass filter).

The scheme is based on the scalar error equation

$$e = (\tilde{D}-D) y_1 + (\tilde{N}-N) u_1. \quad (8.1)$$

If the parameter adjustment law is defined as

$$\Delta \alpha_j = -ke(p^j y_1) \quad p \equiv d/dt \quad (8.2)$$

$$\Delta \beta_1 = -ke(p^1 u_1)$$

and the parameter misalignment vector is defined by

$$\underline{\phi}^T = [\Delta \alpha_0, \Delta \alpha_1, \dots, \Delta \alpha_{n-1}, \Delta \beta_0, \dots, \Delta \beta_m],$$

it is possible to show that the function

$$V = \frac{1}{2} \underline{\phi}^T \underline{\phi} \text{ is a Liapunov function with } \dot{V} = \underline{\phi}^T \dot{\underline{\phi}} = -ke^2 \leq 0. \quad (8.3)$$

However  $\dot{V}$  in (8.3) is only negative semidefinite, i.e.  $e$  may be zero for  $\underline{\phi} \neq 0$ , and  $\underline{\phi} = 0$  is required for identification. By application of a theorem due to LASALLE [L2], Lion has shown that global asymptotic stability will be achieved if  $u$  is periodic, and meets certain conditions as to

frequency content.

KUDRA and NARENDRA [K2] have used direct adjustment of the model to solve the identification problem for the multivariable time-variable plant. In contrast to Lion, all plant states must be known. No restrictions are placed on the form of the state equations.

Given the state equations

$$\text{Plant } \dot{\underline{x}} = A^* \underline{x} + B^* r \quad (8.4)$$

$$\text{Model } \dot{\underline{y}} = C \underline{y} + (A-C) \underline{x} + B r$$

where  $C$  is a stability matrix, and  $A, B$  represent the model, then the error equation ( $\underline{e} = \underline{y} - \underline{x}$ ) becomes

$$\dot{\underline{e}} = C \underline{e} + \Phi \underline{x} + \Psi u \quad (8.5)$$

in which  $\Phi = A - A^*$ ,  $\Psi = B - B^*$ . It is seen that  $\underline{e} \rightarrow 0$  if the parameter misalignment matrices approach zero.

Following the scheme outlined in Section VI for a single-input plant, the direct adjustment scheme for a time-variable multi-input plant becomes

$$\dot{\underline{a}}_1 = -R_1 \underline{a}_1 - P \underline{e} x_1 \quad (8.6)$$

$$\dot{\underline{b}}_1 = -S_1 \underline{b}_1 - P \underline{e} r_1$$

where it is observed that (8.6) is a modification of (2.14).  $R_1, S_1$  are any positive definite diagonal matrices, and are introduced so that  $V$  will contain negative definite component in  $\underline{E}$  (See Section VI). It is noted that the model parameters ( $\underline{a}_1, \underline{b}_1$ ), rather than the plant parameters, are adjusted in this case. Computer simulation of fourth-order two-input plant is shown to yield good results.

### IX. Adaptive Observer

When there is ignorance of plant parameters, a LUENBERGER [L6,7,8] observer cannot be constructed. For this reason, an observer that adapts to unknown plant parameters might greatly extend the range of existing control laws.

An adaptive observer of the full-order Luenberger kind for single-input single-output observable linear plants with unknown parameters has been reported by CARROLL and LINDORFF [C1]. In their adaptation, the observer parameters are changed directly by a Liapunov adaptive law requiring no derivatives while maintaining almost complete freedom in the selection of observer eigenvalues. The state of the observer is guaranteed to eventually duplicate the state of the plant under observation whenever the input to the plant contains sufficient frequencies.

A similarity transformation matrix  $T$  always exists (though its elements are usually unknown) for observable systems which transforms the plant matrix  $A^*$  into the "output" form defined by

$$T^{-1} A^* T = \begin{bmatrix} -a_{11} & & & \\ -a_{21} & I & & \\ \vdots & & \ddots & \\ -a_{n1} & 0 & 0 & \dots & 0 \end{bmatrix} = A_0^* \quad (9.1)$$



and, also,  $C^*T=C^*$  where  $C^*$  is the plant output matrix of the form  $[1 \ 0 \ 0 \ \dots \ 0]$ . With the plant thus transformed, the error equation has the form

$$\dot{e} = K_0 e + \Delta B r + \Delta A x + D u \quad (9.2)$$

$$e = z - x$$

when the equation of the observer is

$$\dot{z} = K_0 z + G C^* x + H r + D u \quad (9.3)$$

where  $K_0$  a constant  $n \times n$  matrix in output form of observer eigenvalues,  $D$  a diagonal  $n \times n$  matrix.  $G$ ,  $H$ , and  $T$  are adjusted by adaptation.

(9.3) is collapsed into a scalar equation in  $e_1$  similar to eq.(4.3). Letting

$$p = \frac{d}{dt}, \text{ this equation may be written}$$

$$(p + \lambda_1) \prod_{i=2}^n (p + \lambda_i) e_1 = \prod_{i=2}^n (p + \lambda_i) \left| \sum_{j=0}^{n+m+1} \phi_j v_j \right| - f(\phi_j^{(i)}, v_j^{(k)})$$

$$+ \sum_{i=0}^{n-1} d_i u_1^{(i)}$$

where  $v_i$  are appropriate filtered states as in (4.5). It may be shown that  $\{u_i\}$  can be chosen to eliminate the term  $f(\phi_j^{(i)}, v_j^{(k)})$ , without the use of derivative networks. Then (9.4) is equivalent to:

$$\dot{e}_1 + \lambda_1 e_1 = \sum_{i=0}^{n+m+1} \phi_i v_i. \quad (9.5)$$

This is in a form suitable for direct adaptation. The adaptive law for  $\phi_i$  contains only the variable  $e_1$  which is available.

As a byproduct of adaptation, the matrix  $T$  can be constructed so that  $e_i \rightarrow 0$  implies that the observer state eventually converges to the (untransformed) plant state.

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